Thema
UMR 8184
THéorie Économique, Modélisation et Applications

THEMA Working Paper n°2019-11
Université de Cergy-Pontoise, France

Fundamental Pricing of Utility Tokens

Julien Prat, Vincent Danos, Stefania Marcassa

UNIVERSITÉ de Cergy-Pontoise
CNRS CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

June 2019

Fundamental Pricing of Utility Tokens

Julien Prat,* Vincent Danos† and Stefania Marcassa‡

First version: November 2018
This version: June 2019

Abstract

We explain how to evaluate the fundamental price of utility tokens. Our model endogenizes the velocity of circulation of tokens and yields a pricing formula that is fully microfounded. According to our approach, tokens are valuable because they have to be immediately accessible when the platform service is needed, a requirement that is reminiscent of the cash-in-advance constraint in the theory of money.

KEYWORDS: Asset pricing, utility tokens, Blockchain, ICOs

JEL CODE: G12, G24.

1 Introduction

The vast majority of startups finance their growth by raising equity from venture capitalists. This market dominance has recently been challenged by a new fundraising method that leverages Blockchain technologies. Following the examples of Bitcoin and other cryptocurrencies, such as Ethereum and Ripple, a growing number of startups rely on Initial Coin Offerings (ICOs hereafter) to raise capital: The company issues a new cryptocurrency, and investors receive its "tokens" in exchange for legal tender or other cryptocurrencies. The tokens derive their value from the fact that they will be

*CNRS, CREST and Ecole Polytechnique. Corresponding author: julien.prat@ensae.fr
†CNRS, ENS-Ulm.
‡University of Cergy-Pontoise.

used to purchase the goods or services offered by the issuer once its platform becomes operational. The value of the token is therefore expected to increase with the size of the business, thus rewarding early investors.

Although the disruptive potential of ICOs is now widely recognized, their adoption beyond the crypto-community has been hindered by their controversial reputation. The most common criticism is that ICOs are used to draw unsophisticated investors towards nonviable projects. Such practice will only be curbed through the creation of a reliable framework for the valuation of ICOs, making it possible for investors to identify real opportunities.

This paper takes one of the early steps in that direction by proposing a fully micro-founded pricing model for utility tokens. We show why tokens intrinsically differ from other financial instruments, such as debt or equity, and thus cannot be priced using off-the-shelf valuation techniques. Our model identifies the fundamental value of tokens as a function of two sets of primitives: consumers' preferences and technological constraints. It characterizes their price trajectory from the ICO date until convergence to the long-run equilibrium. In particular, our solution endogenizes the evolution of token velocity whereas reduced-form pricing techniques currently used by investors arbitrarily specify the speed at which tokens circulate.

Relying on a formal model allows us to clarify the answers to the following essential questions. Why and under which conditions are tokens valuable? What is the actual cost of ICOs for the issuer?

Our response to the first question is that tokens are valuable to the extent that, when needed, the platform has to be accessed immediately. In other words, users cannot delay their consumption until they have been able to acquire the required tokens in the secondary market. This constraint is reminiscent of the cash-in-advance constraint commonly advocated to endow cash with some intrinsic value. The parallel is not really surprising since, after all, tokens are an electronic form of money. Their interesting specificity with respect to cash is that the token-in-advance constraint can be hardwired into the technological specification of the platform, and is also likely to depend on the type of services provided by the company.[1]

Provided that the token-in-advance constraint holds, tokens are valuable and we derive a pricing formula which only depends on the preferences of users. It shows that services are sold at a price that is *below* their marginal utility. Quite intuitively,

[1]In practice, most platforms combine staking incentives and lock-in periods to slow down the circulation of tokens.

users have to be compensated for holding tokens that do not bear any interest. Such incentives are provided at the time of trade by ensuring that users extract some convenience yield from the exchange of the marginal token. But this implies that services are sold at a discount, as the equilibrium price is smaller than the one that would have prevailed if services could be bought with fiat money. This discount is the implicit cost of relying on an ICO instead of venture capital to finance early growth. By issuing utility tokens, the company commits to selling its product at a discounted price in the future. This insight clarifies the often muddled debate over the trade-off between ICOs and equity financing, most notably by dispelling the too widely shared belief that ICOs are a free lunch for issuers.[2]

Related literature. ICOs being a very recent phenomenon, the related academic literature is still in its infancy.[3] The first generation of papers focused on the value of privately-issued digital currencies. Athey et al. (2016) analyze the determinants of their exchange rates, demonstrating that investors may hoard currencies in anticipation of future transactional usage. A similar mechanism is at work in the dynamic version of our model where most tokens are initially held by investors. A related strand of research revisits the indeterminacy of exchange rates between two currencies originally established by Kareken and Wallace (1981). Garratt and Wallace (2008) distinguish the central bank from the privately issued currency by introducing a storage cost for the former and a disaster risk for the later. Pagnotta (2018) explicitly models how the crash risk is determined by miners' investment, thus giving rise to price–security feedback loops that can amplify or dampen the impact of demand shocks on Bitcoin price. Uhlig and Schilling (2018) show that indeterminacy can support a speculative equilibrium where the cryptocurrency is held in anticipation of its appreciation. Biais et al. (2018) embed a dual-currency regime into an OLG model and show how their framework can be taken to the data. Our paper differs from this literature in that we are not considering cryptocurrencies whose purpose is to serve as a universal means of payment, but instead utility tokens whose detention is required to consume a particular product. Hence our pricing formula is directly derived from consumers preferences rather than from transactional benefits.

[2]It is also sometimes argued that ICOs are costly because they amount to selling for free the amount of services corresponding to the mass of issued tokens. But this argument is misleading as the company can always sell back its tokens on the secondary market. The loss therefore occurs at the pricing margin, through downwards adjustments, and not at the quantity margin, through lost sales.

[3]The first documented token sale was held by Mastercoin in 2013.

Our paper is therefore more closely related to the growing literature studying ICOs. A first branch focuses on corporate finance issues related to the incentives of investors and entrepreneurs. Catalini and Gans (2018) show that ICOs may be more efficient than venture capital when participants in the ICO market are well informed, as token prices reveal the actual quality of the project to a wider set of investors. Chod and Lyandres (2018) explains why token sales lead to underinvestment because they generate an agency conflict between the entrepreneur and investors. In spite of this drawback, they find that ICOs can dominate traditional venture capital when investors are underdiversified. Canidio (2018) also underlines the agency conflicts induced by ICOs since there is a non negligible probability that the entrepreneur will sell all her tokens and halt the development of her project. Moreover, even when this risk is avoided, the entrepreneur will behave myopically by maximizing the project value in the post-ICO period and not over its all lifetime. A more positive strand of paper outlines the coordination benefits of ICOs in applications with network effects. Bakos and Halaburda (2018) and Lia and Mann (2018) show that token sales may help overcome coordination failure, since token sales provide a signal about consumers' willingness to use the platform.

In order to focus on the pricing of tokens, we abstract from issues related to incentives alignment between entrepreneurs and investors. In this respect, the paper most closely related to ours is Cong et al. (2018). They also derive a dynamic asset pricing model of tokens, showing that token appreciation can accelerate platform adoption by allowing users to partially internalize network externalities. The main difference between our approaches is that we study utility tokens that have to be exchanged in order to access the platform, whereas Cong et al. (2018) assume that tokens give access to a stream of services when they are staked. As a result, the velocity of circulation is not a relevant statistics in their model because tokens are always held by users. By contrast, the share of tokens held by investors is endogenously determined in our model, and its evolution drives changes in the velocity of circulation, thus explaining why this statistics has been the subject of intense scrutiny in the crypto-community.

Structure of the paper. Section 2 derives the equilibrium price of tokens in steady-state. Section 3 shows how this price can be used to finetune the amount of tokens issued at the ICO stage. Section 4 describes how token velocity evolves over time by extending our framework to a setup with gradual adoption. Proofs of claims and propositions are relegated to the Appendix A.1.

2 Steady-State Solution

2.1 Model Set-Up

We consider a platform which issues tokens to finance its development. Tokens are valuable because they allow their owners to purchase the goods and services produced by the platform. The overall supply of tokens, or monetary base, is equal to M. To simplify matters, we assume that the mass of tokens remains constant over time, which is actually true for most ICOs since they impose an upper-bound on the supply of tokens issued.

There are two markets: (i) a trading market where tokens are bought using fiat money, and (ii) a commodity market where tokens are sold in exchange of the platform's output. The platform has monopoly power on the commodity market and *commit to exchanging* one unit of service against each token. The price or exchange rate of the token in fiat money is denoted by p_t. It is determined on the perfectly competitive and frictionless trading market.

We normalize the mass of users to one. In each period, a *constant share* $\lambda \in (0,1)$ of users are willing to consume the platform's services. Then they derive utility $u\left(c\right)$ from consuming c units of service, where $u\left(c\right)$ is a standard utility function ($u'(c) > 0, u''(c) < 0, \lim_{c\to\infty} u'(c) = 0$). Hence the per-period utility function of user $i \in [0,1]$ reads

$$\mathcal{U}\left(c, d^i\right) = u(c) * d^i, \text{ where } d^i = \begin{cases} 0 \text{ with probability } 1-\lambda \\ 1 \text{ with probability } \lambda \end{cases}. \tag{1}$$

2.2 Equilibrium Price

Each period is divided into two sub-periods. As summarized in Fig. 1, the commodity market opens first and preference shocks d^i are revealed. Users can buy the service only if they have entered the period with some tokens. Then the commodity market closes and the trading market opens, allowing users to rebalance their token holdings by selling and buying tokens at the market price p_t. The timing is crucial. Suppose instead that users first observe their willingness to consume and then adjust their token holdings. Since tokens do not bear any interest, users would find it optimal to hold zero tokens at the beginning of the period and the market price p_t would collapse to zero. The only use of tokens is that they enable consumers to satisfy their needs.

Thus they are valuable because the service is needed immediately and it is too costly to wait for the next period.

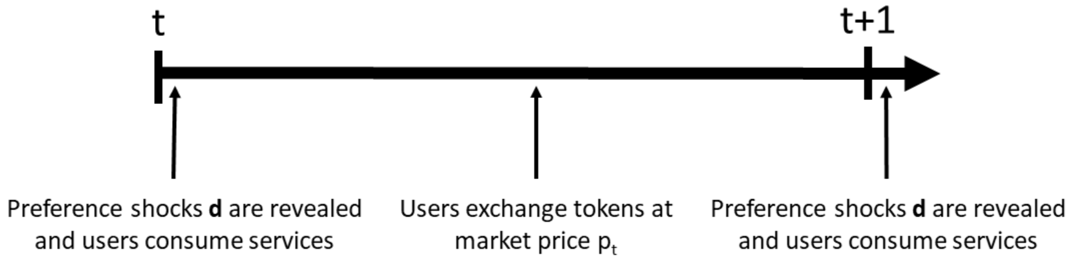

Figure 1: TIMING ASSUMPTIONS.

When the trading market opens, users decide how many tokens m to carry into the next period. Since users can instead invest their money at the risk-free rate r, their optimal returns read

$$v(p_t, p_{t+1}) \equiv \max_{m \geq 0} \left\{ \max_{c \in [0,m]} \{\lambda [u(c) + p_{t+1}(m-c)]\} + (1-\lambda)p_{t+1}m - (1+r)p_t m \right\}. \quad (2)$$

The value of the dummy variable d will be drawn in the following period, where it will be equal to 0 with probability $1-\lambda$. Then the agent will not need the service and so she will enter next period's trading market with the same amount of tokens m, thus earning a reward equal to $p_{t+1}m$, as indicated by the penultimate term in (2). With the complementary probability λ, the dummy variable d will be equal to 1 and the agent will value the services provided by the platform. Then she will choose her optimal level of consumption under the constraint $c \in [0, m]$ because consumption can never be greater than token holdings. If the agent does not consume all her tokens, she will enter next period's trading market with $m-c$ tokens, thus earning the financial reward $p_{t+1}(m-c)$ on top of the utility benefits $u(c)$. Finally, we ensure that v measures the optimal net returns by subtracting the value that would have been obtained if the funds $p_t m$ had been invested at the risk-free rate r.

For the returns function v to be well defined, the token has to appreciate at a rate that is lower than the risk-free rate, as otherwise agents would find it optimal to hoard an infinite amount of tokens. This requirement also ensures that the constraint $c_t \leq m_t$ always binds for users that wish to access the platform's services. This is because consumption is determined after the value of the demand shock d has been revealed, whereas token holdings are decided beforehand. Given that agents take their investment decision behind the veil of ignorance, they face the risk of not needing the service. This is why consumers are always rationed by the amount of tokens they carry from one period to the next.[4] The returns function is therefore equivalent to

$$v(p_t, p_{t+1}) = \max_{m \geq 0} \{\lambda u(m) + (1-\lambda)p_{t+1}m - (1+r)p_t m\}, \tag{3}$$

and token holdings are optimal when

$$rp_t = \underbrace{\lambda [u'(m^*) - p_{t+1}]}_{\text{Convenience Yield}} + \underbrace{p_{t+1} - p_t}_{\text{Capital Gain}}. \tag{4}$$

The rate of return on tokens can be decomposed into two components: a capital gain and a convenience yield. The capital gain is standard since it corresponds to the appreciation in the price of the token. By contrast, the convenience yield is specific to utility tokens. The marginal token can provide a service whose utility is equal to $u'(m^*)$. But the service is delivered *in exchange* of the token. Thus one also has to take into account the loss of the token and deduct its price from the marginal benefit. From the standpoint of pricing theory, this is the main difference between tokens and shares. Since shares do not have to be exchanged to provide their owners with dividends, their fundamental value is equal to the discounted sum of all future dividends. Utility tokens, on the other hand, do not yield any benefits if they are not traded, so their fundamental value is equal to the *discounted surplus of the next trade*. In our model, a trade occurs with probability λ, which explains why the surplus $u'(m^*) - p_{t+1}$ is multiplied by λ in the expression of the convenience yield.

All agents being identical, they hoard the same amount of tokens. Since the mass of users is normalized to one, the market for tokens clears when

$$m_t^{i,*} = M \text{ for all } t \text{ and all } i \in [0,1]. \tag{5}$$

Replacing the market clearing condition into (4), we find that the price of tokens obeys the following law of motion

$$p_t = \frac{1}{1+r}[\lambda u'(M) + (1-\lambda)p_{t+1}]. \tag{6}$$

[4]This result immediately follows comparing the FOCs for consumption, $u'(c^*) = p_{t+1}$, with the one for token holdings (4). Since we focus on cases where $(1+r)p_t > p_{t+1}$, $u'(m^*) > p_{t+1} = u'(c^*)$ and so the feasibility constraint, $c \leq m^*$, binds.

Setting p_{t+1} equal to p_t, yields the following solution for the steady-state price

$$\hat{p} = \frac{\lambda}{r + \lambda} u'(M) < u'(M). \tag{7}$$

As any other initial condition than $\hat{p}$ generates diverging trajectories for p_t, the steady-state $\hat{p}$ is also the unique equilibrium price. Not surprisingly, $\hat{p}$ is decreasing in token supply M. More interestingly, services are paid at a price that is *lower* than their marginal utility. This is the cost involved in requiring users to pay in tokens as the equilibrium price is smaller than the one that would have prevailed if services could be bought using fiat money. This implicit discount compensates users for the lost interests and is therefore proportional to the risk-free rate, which explains why $\hat{p}$ converges to $u'(M)$ when r goes to zero.

2.3 Endogenous User Base

The equilibrium price ensures that the trading market clears. When users are homogenous, as in the previous subsection, market clearing implies that potential demand is saturated. By contrast, when users are heterogenous, the user base becomes endogenous. For simplicity, we assume that users share the same per-period utility function (1), but incur different fixed costs of accessing the platform. These fixed costs are *inversely proportional* to the level of technological expertise χ_i of user i. The parameter χ_i captures the opportunity cost of the time devoted to using the platform. User i finds it optimal to hold some tokens whenever

$$v(p_t, p_{t+1}) - \frac{1}{\chi_i} \geq 0.$$

Since we focus on the steady-state, we simplify our notation by introducing $\hat{v}(p) \equiv v(p,p)$ to denote returns when the price remains constant. Potential users draw their ability from the distribution $G(\chi)$. Thus the user base in steady-state, which we denote by $\hat{N}$, is equal to

$$\hat{N} = 1 - G\left(\frac{1}{\hat{v}\,(p)}\right). \tag{8}$$

Since $\hat{v}(p)$ is decreasing in p, (8) defines a decreasing relation between $\hat{N}$ and p. Intuitively, less agents access the platform when the price of its service goes up.

The equilibrium price $\hat{p}$ is obtained interacting this condition with the law of motion for p_t. First, we have to adjust the market clearing condition by rescaling the overall mass of tokens by the number of users

$$m_t^{i,*} = \frac{M}{N_t} \text{ for all } t \text{ and all } i \in [0,1]\,. \tag{9}$$

Then the law of motion (6) generalizes to

$$p_t = \frac{1}{1+r}\left[\lambda u'\left(\frac{M}{N_t}\right) + (1-\lambda)p_{t+1}\right],$$

so that the price can be stable solely if

$$p = \frac{\lambda}{r+\lambda} u'\left(\frac{M}{\hat{N}}\right). \tag{10}$$

Equation (10) defines an increasing relation between $\hat{N}$ and p. As the supply of tokens M is fixed, when the price increases, the share of users will increase, but each of them will hold a lower amount of tokens.

Interacting it with (8) yields a system of two equations in two unknowns, $\hat{N}$ and p, with at most a unique solution.

A Closed-Form Solution. The model can be solved analytically when the utility function of users is CRRA and when their abilities are sampled from a Pareto distribution

$$H1: u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \text{with } \sigma \in (0,1)\,,$$
$$H2: G(\chi) = \max\left\{0, 1-\left(\frac{\underline{\chi}}{\chi}\right)^{\alpha}\right\} \text{ for all } \chi > 0, \text{ with } \underline{\chi} > 0 \text{ and } \alpha > 0.$$

Although restrictive, both hypotheses have some empirical support: CRRA is among the most common utility specification; whereas models with heterogeneous agents usually rely on Pareto distributions to capture fat tails in the distribution of abilities. From a formal standpoint, Assumption H1 enables us to explicitly derive net returns as a function of the steady-state price. Combining this solution with H2 yields a closed-form expression for the equilibrium condition (8), defining a locus for the mass of users $\hat{N}$ that is weakly decreasing in p. Interacting this condition with the rest point requirement (10) yields a system of two equations for the two unknowns $\hat{N}$ and $\hat{p}$.

Claim 1 *When H1 and H2 hold, the equilibrium mass of users $\hat{N}$ and token price $\hat{p}$ are uniquely determined by the following system of equations*

$$\textit{Participation constraint:} \quad \hat{N} = \min\left\{1, \left[\frac{\underline{\chi}\lambda^{1+\rho}}{\rho\left[(r+\lambda)\,\hat{p}\right]^{\rho}}\right]^{\alpha}\right\}, \tag{11}$$

$$\textit{Demand for tokens:} \quad \hat{N} = M\left[\frac{\hat{p}\,(r+\lambda)}{\lambda}\right]^{\frac{1}{\sigma}}, \tag{12}$$

where $\rho \equiv (1-\sigma)/\sigma$.

Proof. See Appendix A.1.1. ■

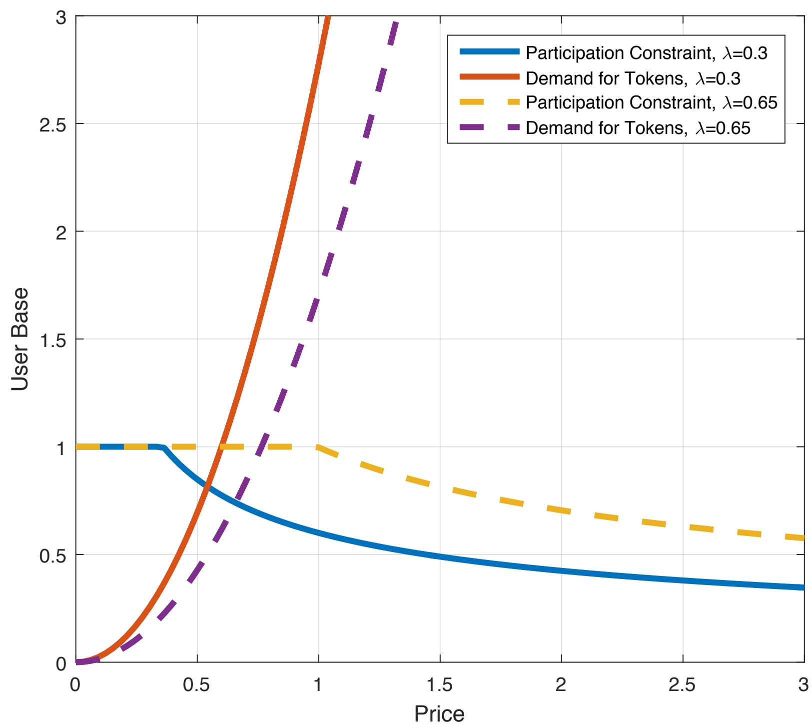

Figure 2: EQUILIBRIUM PRICE AND USER BASE. PARAMETERS: $r = .05$, $\sigma = .5$, $\kappa = 1$, $M = 1$, $\overline{\chi} = 2$.

Figure 2 illustrates how the equilibrium price and market size are pinned down by the conditions (11) and (12) for two values of λ, i.e. the probability that a trade occurs. As λ increases the participation constraint shifts upwards: for a given price level, more users will need the service. The demand for tokens (M/N) is increasing in the mass of users: as the supply of tokens is fixed, when the mass of users goes up, individual holdings goes down, thus sustaining a price increase. As the probability of transaction λ goes up, so does the price. The solid lines ($\lambda = 0.3$) represent an equilibrium where the market is not saturated since they interesect at a level of the user base which is lower than one. The dotted lines ($\lambda = 0.65$) illustrate an equilibrium where the market is saturated, where all the users consume the service. Hence, a higher probability of trade is associated with a higher equilibrium price and a bigger mass of service users.

3 ICO Design

3.1 Platform's Profits

The platform incurs a cost per period $\kappa(C)$ that is proportional to the overall consumption of its output $C \equiv \int_0^1 c_i di$. In steady state, $\hat{C} = \lambda\hat{c} = \lambda M$, and so profits π stabilize at the following level

$$\begin{aligned}\hat{\pi}(M) &= \hat{p}\hat{C} - \kappa(\hat{C}) \\ &= \frac{\lambda}{r+\lambda}u'(M)\lambda M - \kappa(\lambda M).\end{aligned}$$

Asking for the service to be paid in tokens lowers equilibrium profits since the service is sold at a price $\hat{p}$ that is smaller than its marginal utility $u'(M)$.

Claim 2 *Assume that: (i) H1 holds, i.e.* $u(c) = c^{1-\sigma}/(1-\sigma)$*; (ii) Users are homogenous; (iii) marginal costs are constant, so that* $\kappa(\hat{C}) = \kappa\hat{C}$*. Then the token mass* $M^*(\lambda)$ *that maximizes profits as a function of* λ *is well defined whenever* $\sigma \in (0,1)$ *and is equal to*

$$M^*(\lambda) = \left[\frac{\lambda(1-\sigma)}{\kappa(r+\lambda)}\right]^{\frac{1}{\sigma}}.$$

Figure 3 reports the equilibrium price and platform's profits as a function of the overall mass of tokens when, as in Claim 2, users' utility is CRRA and marginal costs are constant. The upper-panel shows that prices are decreasing in M, while the lower-panel shows that profits are globally concave with a global maximum. We use three different values of λ to compute the equilibrium schedules. Not surprisingly, price and profits are increasing in the frequency λ at which users need to use the services provided by the platform. Accordingly, the optimal mass of tokens $M^*(\lambda)$ is also increasing in λ.

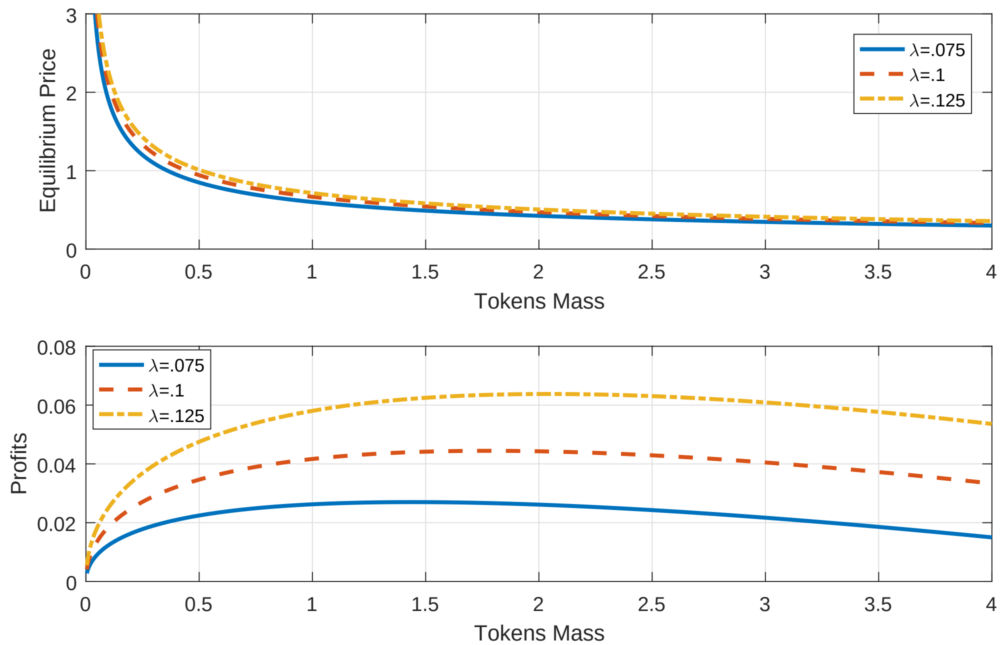

Figure 3: EQUILIBRIUM PRICE AND PROFITS AS A FUNCTION OF M FOR DIFFERENT VALUES OF λ. PARAMETERS: $r = .05$, $\sigma = .5$, $\kappa = .25$.

3.2 Optimal Token Supply

We have characterized the production stage. Adding an initial period, where the platform sets in advance the amount of tokens to be issued, allows us to model the optimal policy at the ICO stage. Token supply is decided behind the veil of ignorance. For instance, platform owners are likely to be uncertain about the actual share λ of users that will need their services in each period. Hence they should choose the overall supply of tokens M^* that maximizes the following objective function

$$M^* = \arg\max_{M>0} \left\{ \int \hat{\pi}\left(M|\,\lambda\right) d\phi\left(\lambda\right) \right\}, \tag{13}$$

where $\phi\left(\lambda\right)$ denotes the owners' prior about λ.

As shown in Figure 4, there exists a unique solution to problem (13), when the conditions in Claim 2 are satisfied and λ is sampled from a lognormal distribution.

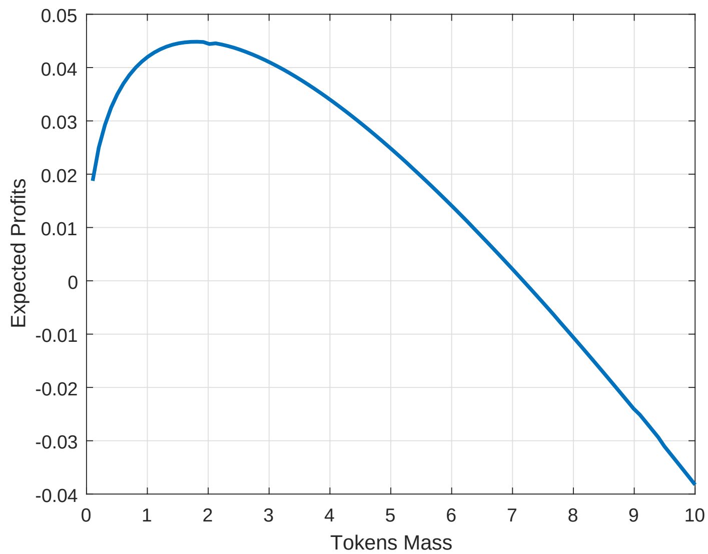

Figure 4: EXPECTED PROFITS AT THE ICO STAGE. PARAMETERS: $r = .05$, $\sigma = .5$, $\kappa = .25$, $\log(\lambda) \sim N(\log(.1) - .02, .2)$

4 Gradual Adoption

We now explain how to model the evolution of the token price from the ICO date until convergence to the long run steady-state. This transition might take a while as users gradually migrate to the platform. Slow adoption can be due to a variety of reasons ranging from reputation building and growing awareness about the services provided by the platform, to improvements in the underlying technology. We adopt the last view and focus on cases where user adoption builds up over time because the platform becomes more and more efficient. We capture technological progress through the introduction of the demand shifter z. The quality of the services provided by the platform is proportional to z as

$$u(c; z) = zu(c; 1). \tag{14}$$

To ease notation, we hereafter refer to $u(c; 1)$ as $u(c)$. We also devise our model in continuous time because it greatly simplifies the analysis. The continuous time counterpart to equation (3) reads

$$v\left(p_t, \dot{p}_t, z_t\right) = \max_m \left\{\lambda\left[z_t u\left(m\right) - p_t m\right] + \dot{p}_t m - r p_t m\right\}. \tag{15}$$

where λ now denotes the Poisson rate at which users need to access the platform.[5]

As in Section 2.3, user i draws her ability from the distribution G, and buys tokens when net returns exceed her fixed costs, i.e. when $v\left(p_t, \dot{p}_t, z_t\right) \geq \chi_i^{-1}$. Hence market size N at time t is a function of the vector $\left(p_t, \dot{p}_t, z_t\right)$ that satisfies

$$N\left(p_t, \dot{p}_t, z_t\right) = 1 - G\left(\frac{1}{v\left(p_t, \dot{p}_t, z_t\right)}\right). \tag{16}$$

Price dynamics. The rate at which tokens appreciate depends on whether the marginal holder is a user or an investor. If overall demand from users is too low to clear the market, i.e. $M > N_t m_t^*$, the marginal token will be held by agents that speculate on its appreciation. Then, assuming an infinitely elastic supply of capital from investors, the price has to grow at the rate of interest so that[6]

$$\dot{p}_t = r p_t, \text{ when } m_t^* < \frac{M}{N_t}. \tag{17}$$

In this regime, the token behaves as an asset bubble because it has no convenience yield for the marginal holder. By contrast, when the demand from users clears the market, i.e. $M = N_t m_t^*$,[7] price dynamics is governed by users' optimality condition so that[8]

$$\dot{p}_t = -\lambda\left(z_t u'\left(\frac{M}{N\left(p_t, \dot{p}_t, z_t\right)}\right) - p_t\right) + r p_t, \text{ when } m_t^* = \frac{M}{N_t}. \tag{18}$$

Since this equation has the exact same structure as its discrete time counterpart (4), we refer to Section 2.2 for a discussion of the convenience yield $\lambda\left(z_t u'\left(\frac{M}{N_t}\right) - p_t\right)$.

Equilibrium path. The evolution of the token price is driven by changes in the produc-

[5]See Appendix A.2.2 for a formal derivation.

[6]Equation (17) also arises as a limit case of (18) when $\lambda = 0$, that is when the platform's services will never be needed, as would be the case for pure investors.

[7]Note that there is no need to consider regimes where $N_t m_t^* > M$ because $\dot{p}_t$ is a free variable that will adjust, namely decrease, until token demand from users is again equal to token supply.

[8]Equation (18) is the FOC of (15) where m_t^* has been replaced by M/N_t to take into account the market clearing condition. See also eq. (30) in Appendix A.2.2 for a derivation of (18) when users are homogenous.

tivity parameter z. We simplify the analysis by assuming that z follows a deterministic trajectory that is commonly known. We further restrict our attention to processes that increase over time and converge to a finite limit[9]

$$H3: z_t \text{ follows a deterministic process such that } \dot{z}_t \geq 0 \text{ and } \lim_{t\to\infty} z_t = \bar{z} < \infty.$$

When productivity is low, user adoption is also low. Then most tokens are held by investors and their price grows at the rate of interest, as specified in (17). But this is obviously not sustainable in the long-run as the price would have to diverge to infinity. Thus there must exist a productivity level above which all tokens are held by users. In this regime the law of motion of p is governed by (18). Hence the equilibrium path is pinned down by the boundary condition when time goes to infinity. Since the rational trajectory must be such that all tokens are held by users in the long-run, the equilibrium price converges to the steady-state analyzed in Section 2.3, i.e. $lim_{t\to\infty} p_t = \hat{p}$ where

$$\hat{p} = \frac{\lambda}{r+\lambda}\bar{z}u'\left(\frac{M}{1-G\left(\frac{1}{v(\hat{p},0,\bar{z})}\right)}\right). \tag{19}$$

The terminal condition anchors the whole equilibrium path and, in particular, the initial price p_0 as no other initial condition generate trajectories that converge $\hat{p}$. The equilibrium path can therefore be solved using a shooting algorithm which checks, at each step, whether or not $m_t^* N_t < M$, and then chooses the appropriate law of motion between (17) and (18). More precisely, to establish whether the marginal holder is a user or an investor, we first use the fact that $\dot{p}_t \leq rp_t$. Thus the optimality condition for m_t implies that $m_t^* \leq u'^{-1}(p_t/z_t)$. Since flow returns $v(p_t, \dot{p}_t, z_t)$ are increasing in $\dot{p}_t$, it also follows from the market clearing condition (16) that $N_t \leq N(p_t, rp_t, z_t)$. Combining these two upper-bounds, we find that

$$m_t^* N_t \leq \overline{M}(p_t, z_t) \text{ where } \overline{M}(p_t, z_t) \equiv u'^{-1}(p_t/z_t)\, N(p_t, rp_t, z_t). \tag{20}$$

When $\overline{M}(p_t, z_t) < M$, we can conclude that $\dot{p}_t = rp_t$ because the marginal holder is necessarily an investor. By contrast, when $\overline{M}(p_t, z_t) > M$, all holders are users and the rate of appreciation $\dot{p}_t$ is adjusted downwards until market clearing holds. We

[9]Although it is not complicated to let z diverge to infinity, we rule out this possibility because it would muddle the discussion of the boundary conditions without adding new insights.

show below, using a parametric example, that condition (16) generates a locus that neatly separates the (p, z) space into two non-overlapping regions. The structure of the equilibrium is summarized in Definition 3.

Definition 3 *A Markov equilibrium with state variable z_t is a solution such that:*

- *Users hold the amount of tokens m_t^* which maximizes their net returns as defined in eq.* (15)*;*
- *The user base N_t results from optimal participation decisions as defined in eq.* (16)*;*
- *The law of motion of p_t solves the system of first-order differential equations* $(17) - (18)$*, subject to the boundary condition $lim_{t\to\infty} p_t = \hat{p}$ where $\hat{p}$ solves* (19).

A tractable example. In general, the ODE (18) cannot be expressed analytically. However, if we impose H1 and H2, (18) greatly simplifies since it becomes linear. To see why, notice first that when the utility function of users is CRRA, as stated in H1, optimal token holdings read

$$m_t^* = \left[\frac{z_t \lambda}{(r + \lambda) p_t - \dot{p}_t}\right]^{\frac{1}{\sigma}}. \tag{21}$$

This equation is similar to the one prevailing in steady-state but for the inclusion of $\dot{p}_t$. Quite intuitively, an increase in the appreciation rate raises token holdings since it partially compensates users who do not need to access the platform in the current period. Combining this expression with assumption H2 according to which abilities are Pareto distributed, we obtain the linear equations (22) reported in Proposition 1.

Proposition 1 *When H1 and H2 hold, the ODE (18) is linear and reads*

$$\dot{p}_t = \begin{cases} (r + \lambda) p_t - \left[\frac{(z_t \lambda)^{1+\alpha} \left(\underline{\chi}/\rho\right)^{\alpha\sigma}}{M^\sigma}\right]^{\frac{1}{1+\alpha(1-\sigma)}}, & \textit{when } N_t < 1, \\ (r + \lambda) p_t - \lambda z_t u'(M), & \textit{when } N_t = 1. \end{cases} \tag{22}$$

Illustration. We now use a parametric example to illustrate how the equilibrium path is computed. First we need to specify how productivity z_t changes over time. As commonly done in the literature on product diffusion, we assume that its evolution is governed by a logistic curve, so that

$$z_t = \frac{z_0 e^{g_z t}}{1 + z_0 \left(e^{g_z t} - 1\right)}.$$

The values of the diffusion parameter g_z and the starting productivity z_0 are common knowledge. A typical adoption curve is illustrated in Figure 5. For our arbitrary choice of parameters, z_t reaches 99% of its long-run value $\bar{z} = 1$ around the ten years mark.

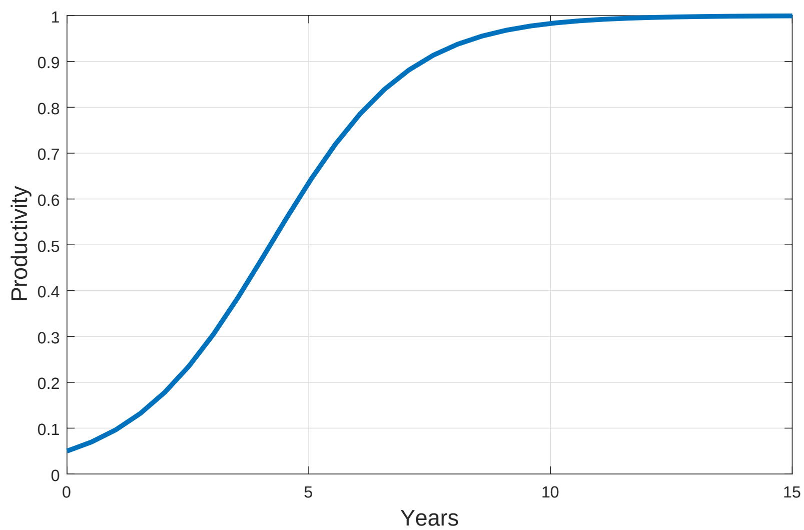

Figure 5: DIFFUSION OF PRODUCTIVITY z_t OVER TIME. PARAMETERS: $z_t = z_0 \exp\left(g_z t\right)/(1 + z_0(\exp\left(g_z t\right) - 1))$, $z_0 = .05$, $g_z = .7$.

In order to compute the equilibrium price path, we need to know which law of motions between (17) and (18) applies at each point in time. In other words, we have to determine whether the marginal token holder is a user or an investor. Computing the values of $\overline{M}\left(p_t, z_t\right)$ defined in (20), we find that, when H1 and H2 hold, $\overline{M}$ separates the (p, z) plane into two non-overlapping regions. As shown in Figure 6, for each level of productivity, there exists a cutoff price below which all tokens are held by users.

Figure 6 indicates that condition (20) allows us to select the relevant ODE for any combination of price and productivity levels. Thus we can use a shooting algorithm to identify the starting price p_0 that generates the unique path which converges to the steady-state $\hat{p}$.

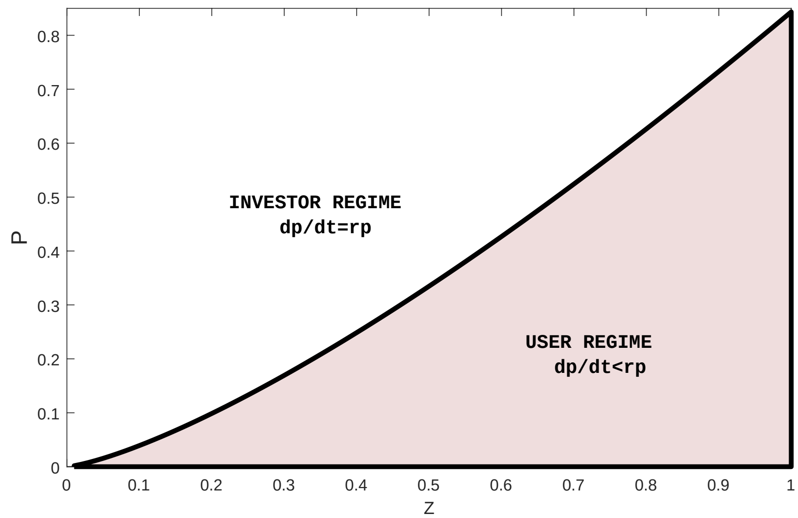

Figure 6: MARGINAL TOKEN HOLDER AS A FUNCTION OF PRICE AND PRODUCTIVITY. PARAMETERS: $\lambda = .3$, $r = .2$, $M = 1$, $\sigma = .5$, $\alpha = 1$, $\underline{\chi} = 2$, $z_0 = .05$, $g_z = .7$.

The path resulting from our choice of parameters is reported in the upper-panel of Fig. 7. The lower panel reports the share of tokens that are held only for speculation purpose. As time goes by, more and more tokens are held with the objective of being used. In our arbitrary example, all tokens are held by users after 4 years and a half. This switch of regime generates an inflexion in the derivative of the price path as the marginal holder now enjoys some convenience yield.

The lower-panel of Fig. 7 implies that the average velocity, $\lambda m_t N_t$, at which tokens circulate evolves over time since those that are hoarded by investors are never exchanged for transaction purposes. As shown in Figure 8, the velocity is initially very low and it gradually converges to λ, i.e. the rate at which users need to access the platform. That is, only users exchange tokens, and they exchange their entire holdings m_t with probability λ and do not trade with probability $1 - \lambda$. Hence, along the transition path, one cannot exogenously set the velocity in order to determine the equilibrium price because both variables are jointly determined.

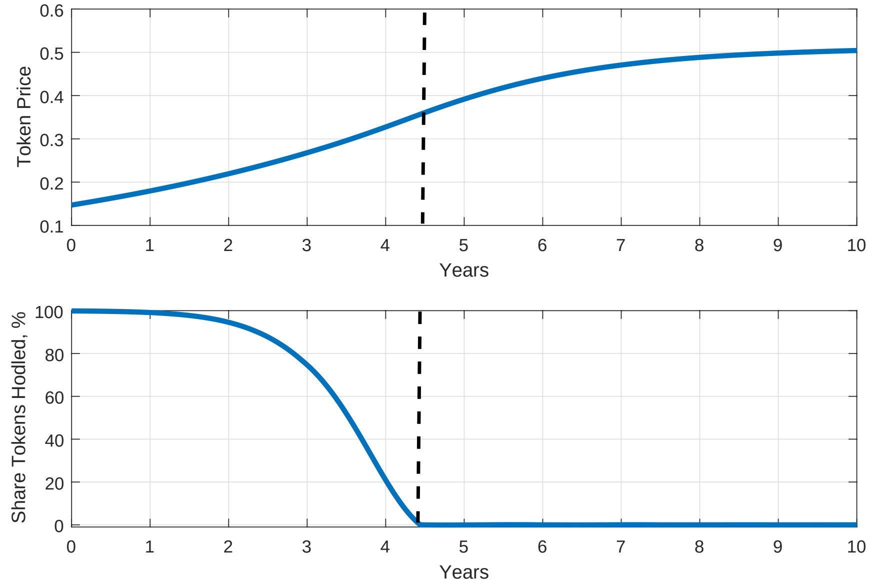

Figure 7: Token Price and Share of Investors as a Function of Time. Parameters: $\lambda = .3$, $r = .2$, $M = 1$, $\sigma = .5$, $\alpha = 1$, $\underline{\chi} = 2$, $z_0 = .05$, $g_z = .7$.

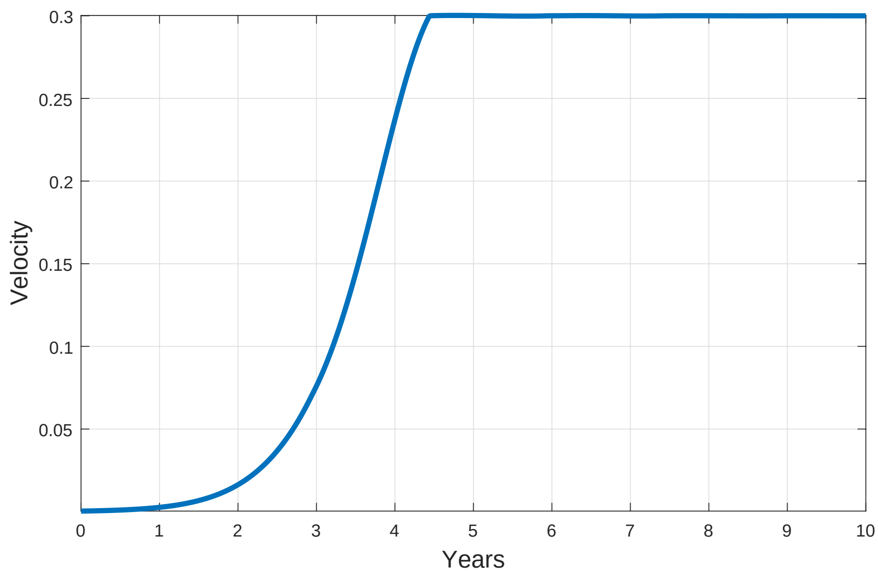

Figure 8: Velocity of Circulation of Tokens. Parameters: $\lambda = .3$, $r = .2$, $M = 1$, $\sigma = .5$, $\alpha = 1$, $\underline{\chi} = 2$, $z_0 = .05$, $g_z = .7$.

Discussion. Before concluding, we outline how our approach to token pricing complements the one proposed by Cong et al. (2018). They assume that users derive a utility flow from holding tokens while we view tokens as pure media of exchange. However, as explained in Appendix A of Cong et al. (2018), this difference is not as fundamental as it may seem. More precisely, they show that their formulation holds when tokens are used as means of payment to save on transaction costs, and transactions are uncertain and lumpy. Hence the main differences between our models lie in the way transaction benefits are modeled and how the equilibrium is determined.

Cong et al. (2018) depart from our set-up in at least two important dimensions. First, they assume that tokens do not have to be explicitly traded to generate utility. Second, they consider that the benefits are proportional to the numéraire value of the tokens.[10] Importing their assumptions into our framework would imply that, instead of solving (15), agents face the following problem

$$v\left(p_t, \dot{p}_t, z_t\right) = \max_m \left\{ z_t u\left(p_t m\right) + \dot{p}_t m - r p_t m \right\}. \tag{23}$$

Utility is now proportional to the market value of token holdings $p_t m$ instead of m. Moreover, since benefits accrue at a constant rate, there is no parameter λ measuring the rate at which transactions are completed. But then again, this is only a formal difference since it amounts to a rescaling of the utility function. There is, however, an essential difference as tokens are not transferred upon the completion of each transaction, which explains why the term $-\lambda m$ is missing on the right-hand side of (23). To see how these two changes fundamentally modify the structure of the model, consider the steady-state solution with homogenous users. The long-run equilibrium associated to (23) is efficient since total revenues in the token and tokenless economies are both equal to $u^{-1}\left(r/\bar{z}\right)$. By contrast, we have shown that the steady-state of our model is strictly dominated by the steady-state of the tokenless economy. The equilibrium price is strictly lower than the one that would prevail if transactions could be settled in cash because users face the risk of not deriving any utility in the current period.

Besides having steady-states with distinct welfare properties, the price trajectories also satisfy different requirements. The market clearing condition in Cong et al. (2018) is fulfilled when all tokens are held by users. We do not impose such a restriction. Instead we allow for a speculative regime where some tokens are held by investors that

[10] Another difference is how Cong et al. (2018) model heterogeneity among users. They assume that users have different flow utility but the same participation cost, precisely the opposite of how we introduce heterogeneity. We do not dwell on this distinction because it only affects the model's algebra and not its main message.

do not enjoy any convenience yield. This prediction is in line with the current state of the market for Blockchain technologies, which has seen relatively few adoptions in spite of an ever-growing valuation.

5 Conclusion

Our model provides an answer to three of the most fundamental questions regarding token pricing; namely, when are tokens valuable, how should they be priced, and what are the costs of raising funds through an ICO? To the first question, our answer is that tokens are valuable when speed is so central to the delivery of the service that users cannot delay its consumption until they have refilled their token holdings. Provided that this requirement holds, our pricing formula highlights that tokens fundamentally differ from other financial instruments because tokens do not generate any dividends until they are exchanged. The equilibrium price is always lower than the marginal utility of the service since prospective users have to be compensated for the opportunity cost of holding tokens instead of interest bearing securities. This in turn clarifies the cost of ICO financing as the platform implicitly commits to selling its product at a discount.

Having a microfounded pricing formula opens up many avenues for future research. Embedding network effects and more sophisticated laws of motion for the demand shifters, as in Cong et al. (2018), would generate richer price dynamics. A more ambitious extension would also endogenize token supply, studying how commitment to some monetary rule could be used to maximize the expected value of the venture.

These lines of investigation are only the first forays into what promises to be a field of research in its own right. Tokenomics is far from providing widely accepted guidelines for the evaluation and design of tokens. As new and more complex tokens are put on the market, the creativity of token issuers is likely to challenge that of researchers for years to come.

References

ATHEY, S., I. PARASHKEVOV, V. SARUKKAI, AND J. XIA (2016): "Bitcoin Pricing, Adoption, and Usage: Theory and Evidence," Stanford University Graduate School of Business Research Paper No. 16-42.

BAKOS, Y. AND H. HALABURDA (2018): "The role of cryptographic tokens and ICOs in fostering platform adoption," Working paper new-york university.

BIAIS, B., C. BISIÈRE, M. BOUVARD, C. CASAMATTA, AND A. J. MENKVELD (2018): "Equilibrium Bitcoin Pricing," Tse working paper, 18-973.

CANIDIO, A. (2018): "Financial Incentives for Open Source Development: the Case of Blockchain," Working paper.

CATALINI, C. AND J. S. GANS (2018): "Initial Coin Offerings and the Value of Crypto Tokens," Working paper national bureau of economic research no. 3137213.

CHOD, J. AND E. LYANDRES (2018): "A theory of ICOs: Diversification, agency, and information asymmetry," Working paper.

CONG, L. W., Y. LI, AND N. WANG (2018): "Tokenomics: Dynamic adoption and valuation," Working Paper Becker Friedman Institute for Research in Economics No. 2018-49.

GARRATT, R. AND N. WALLACE (2008): "Bitcoin 1, Bitcoin 2, ...: An experiment in privately issued outside monies," *Economic Inquiry*, 56(3), 1887–1897.

KAREKEN, J. AND N. WALLACE (1981): "On the Indeterminacy of Equilibrium Exchange Rates," *Quarterly Journal of Economics*, 96(2), 207–222.

LIA, J. AND W. MANN (2018): "Initial coin offering and platform building," Working paper george mason university.

PAGNOTTA, E. (2018): "Bitcoin as Decentralized Money: Prices, Mining Rewards, and Network Security," Working paper.

UHLIG, H. AND L. SCHILLING (2018): "Some Simple Bitcoin Economics," Nber working paper no. 24483.

A Appendix

A.1 Proofs of Claims and Propositions

A.1.1 Proof of Claim 1

Proof. Setting $p_t = p_{t+1}$ in (3) yields

$$\hat{v}(p) = \max_{m \geq 0} \{\lambda u(m) - (r+\lambda)pm\}.$$

Setting again $p_t = p_{t+1}$ in the FOC (4) and using H1, we find that

$$p = \frac{\lambda}{r+\lambda} u'(m^*) = \frac{\lambda}{r+\lambda} \frac{u(m^*)}{m^*} (1-\sigma).$$

Hence net returns in steady-state are equal to

$$\begin{aligned} \hat{v}(p) &= \lambda u(m^*) - (r+\lambda)pm^* \\ &= \lambda \sigma u(m^*) = \frac{\lambda^{1+\rho}}{\rho[(r+\lambda)p]^\rho}, \end{aligned}$$

where $\rho \equiv (1-\sigma)\sigma$. This solution can be reinserted into the first equilibrium condition (8) to obtain

$$\begin{aligned} \hat{N} &= 1 - G\left(\frac{1}{\hat{v}(p)}\right) \\ &= \min\left\{1, \left[\underline{\chi}\hat{v}(p)\right]^\alpha\right\} \\ &= \min\left\{1, \left[\frac{\underline{\chi}\lambda^{1+\rho}}{\rho[(r+\lambda)p]^\rho}\right]^\alpha\right\}, \end{aligned}$$

where the second equality follows from H2. In order to pin down the equilibrium price, a second condition is required. It is provided by the law of motion evaluated at the rest point (10). Replacing $H1$ in (10) yields an upward slopping relation between $\hat{N}$ and p

$$\hat{N} = M\left[\frac{p(r+\lambda)}{\lambda}\right]^{\frac{1}{\sigma}}.$$

■

A.1.2 Proof of Proposition 1

Proof. The FOC of (15) implies that

$$\lambda z_t \frac{\partial u\left(m_t^*\right)}{\partial m} = (r+\lambda)p_t - \dot{p}_t.$$

Hence, as in the steady-state analysis, the net flow returns defined in (15) are equivalent to

$$\begin{aligned} v\left(p_t, \dot{p}_t, z_t\right) &= \lambda z_t u\left(m_t^*\right) - \lambda z_t \frac{\partial u\left(m_t^*\right)}{\partial m} m_t^* \\ &= \lambda z_t \sigma u\left(m_t^*\right) \\ &= \left(\lambda z_t\right)^{\frac{1}{\sigma}} (1/\rho) \left[(r+\lambda) p_t - \dot{p}_t\right]^{-\rho}, \end{aligned} \tag{24}$$

where the second equality holds true because, according to H1, the utility function is CRRA. Assumption H2 allows us to express analytically the share of users holding tokens

$$N_t = 1 - G\left(\frac{1}{v\left(p_t, \dot{p}_t, z_t\right)}\right) = \min\left\{1, \left[\underline{\chi} v\left(p_t, \dot{p}_t, z_t\right)\right]^{\alpha}\right\}.$$

Hence, when $m_t^* = M/N_t$, two configurations may occur.

(i) $N_t < 1$: Then the marginal utility reads

$$\begin{aligned} u'\left(\frac{M}{N_t}\right) = \left(\frac{M}{N_t}\right)^{-\sigma} &= \left(\frac{M}{\left[\underline{\chi} v\left(p_t, \dot{p}_t, z_t\right)\right]^{\alpha}}\right)^{-\sigma} \\ &= \left[z_t \lambda \left(\frac{\underline{\chi}/\rho}{M^{\frac{1}{\alpha}} \left[(r+\lambda) p_t - \dot{p}_t\right]^{\rho}}\right)^{\sigma}\right]^{\alpha}, \end{aligned}$$

where the last equality follows from (24). Reinserting this expression into (18), we find that it is equivalent to

$$\dot{p}_t = (r+\lambda) p_t - \left[\frac{\left(z_t\lambda\right)^{1+\alpha} \left(\underline{\chi}/\rho\right)^{\alpha\sigma}}{M^{\sigma}}\right]^{\frac{1}{1+\alpha(1-\sigma)}}, \text{when } N_t < 1 \text{ and } m_t^* = M/N_t.$$

(ii) $N_t = 1$: Then the marginal utility is by definition equals to $z_t u\left(m_t^*\right)$ and so the law of motion reads

$$\dot{p}_t = (r+\lambda)\,p_t - \lambda z_t u'\,(M)\,, \text{when } N_t = 1 \text{ and } m_t^* = M/N_t.$$

■

A.2 Dynamic Programming Approach

A.2.1 Dynamic Programming Solution in Discrete Time

Let $V_t(m)$ and $W_t(m)$ denote the value function of a user with m units of token just before the first and second sub-periods, respectively. Given that the preference shock is not yet revealed when $V_t(m)$ is evaluated, the value function is by definition equal to

$$\begin{aligned} V_t(m) &= \mathbb{E}\left[\max_{c\in[0,m]} \mathcal{U}\,(c,d) + W_t(m-c)\right] \\ &= \lambda\left[\max_{c\in[0,m]} u(c) + W_t(m-c)\right] + (1-\lambda)W_t(m). \end{aligned} \tag{25}$$

where $\mathbb{E}\,[\cdot]$ is the expectation operator. The constraint $c \in [0,m]$ holds because users cannot consume a quantity of services that is greater than their token holdings m. The dummy variable d is equal to 0 with probability $1-\lambda$. Then the agent does not need the service and so she enters the next sub-period with the same amount of tokens, as indicated by the last term in (25). With the complementary probability λ, the dummy variable d is equal to 1 and the agent values the platform's service. To determine her optimal level of consumption, we need to characterize her continuation value W_t.

The value function $W_t(m)$ at the beginning of the second sub-period satisfies the following Bellman equation

$$W_t(m) = p_t m + \max_{m'}\left\{-p_t m' + \beta V_{t+1}(m')\right\}, \tag{26}$$

where β is the agent's discount factor. The agent can freely rebalance her position at the market price p_t. It follows from (26) that W is linear in m as $W_t'(m) = p_t$. Moreover, the first order condition implies that $V_t'(m_t^*) = p_{t-1}/\beta$, where m_t^* is the optimal amount of tokens by the end of the second sub-period. All agents being identical, they hoard the same amount of tokens. Since we have normalized the mass of users to one, the market for tokens clear when

$$m_t^{i,*} = M \text{ for all } t \text{ and all } i \in [0,1]\,. \tag{27}$$

We now have all the information necessary to differentiate (25) with respect to c. Using the fact that $W_t'(\cdot) = p_t$, we find that optimal consumption is given by

$$c_t^* = \begin{cases} u'^{-1}(p_t) & \text{if } m \geq u'^{-1}(p_t), \\ m & \text{otherwise.} \end{cases}$$

Since there is no uncertainty about p_t, users will carry the minimum amount of tokens necessary for the transaction, so that $m_t^* = c_t^* \leq u'^{-1}(p_t)$. Differentiating (25) with respect to m, and replacing the market clearing condition $m = M$, we finally obtain

$$\frac{p_{t-1}}{\beta} = \lambda u'(M) + (1-\lambda)p_t.$$

Focussing on the steady-state $\hat{p}$, it must hold true that

$$\hat{p} = u'(M)\left[\frac{\beta\lambda}{1-\beta+\beta\lambda}\right] < u'(M). \tag{28}$$

The equilibrium price is decreasing in the overall supply of tokens M, as expected. More interestingly, services are paid at a price that is *lower* than their marginal utility. This is the costs involved in requiring users to pay in tokens as the equilibrium price is smaller than the one that would have prevailed if services could be bought using fiat money. Note however that this implicit discount is proportional to the agent's impatience since $\hat{p}$ converges to $u'(M)$ when β goes to one.

A.2.2 Dynamic Programming Solution in Continuous Time

Devising the model in continuous time alleviates the algebra. We assume that agents are hit by demand shocks that arrive at the Poisson rate λ. As before, conditional on being hit by a demand shock, agents have to buy the service immediately and so cannot go to the trading market to acquire tokens if needed. DISCUSS

Then there is only one value function which satisfies the following Bellman equation

$$rV_t(m_t) = \lambda \max_{c\in[0,m_t]} \{u(c) - p_t\Delta(c)\} + \lambda \max_{\Delta(c)} \{V_t(m_t - c + \Delta(c)) - V_t(m_t)\} + \frac{\partial V_t(m_t)}{\partial t} + [V_t'(m_t) - p_t]\,\dot{m}_t. \tag{29}$$

Given that the cost of marginally increasing the amount of tokens is equal to p_t, holding m_t units can be optimal only if $V_t^{'}(m_t) = p_t$. Hence we can ignore the last term in (29).[11] Moreover, the concavity of the value function implies that it is optimal for agents to restore their token holdings so that $\Delta^*(c) = c$. Thus the Bellman equation consistent with market clearing ($m_t = M$) boils down to

$$rV_t(M) = \lambda \max_{c\in[0,M]} \{u(c) - p_t c\} + \frac{\partial V_t(M)}{\partial t}.$$

Setting token holdings equal to potential demand, $M = c^*$, the first order condition reads

$$rV_t'(M) = \lambda\,(u'(M) - p_t) + \frac{\partial^2 V_t(M)}{\partial t \partial m}.$$

Reinserting $V_t^{'}(M) = p_t$ into this condition, we find that

$$rp_t = \lambda\,(u'(M) - p_t) + \dot{p}_t. \tag{30}$$

At the steady-state, $\dot{p}_t = 0$, and so

$$\hat{p} = \frac{\lambda}{r+\lambda} u'(M) < u'(M). \tag{31}$$

As r is the continuous time counterpart of $(1-\beta)/\beta$, (31) is equivalent to (28). Hence the amount of service is rationed, which generates a trading surplus for users.

[11] In any case, at the steady-state, all tokens are held by users and $\dot{m}_t = 0$.